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Parallel lines without a 1-to-1 correspondence with continuous bodies, and points with a 1-to-1 correspondence with continuous bodies
Mysteries of parallel discrete continuous dotted lines

0. Introduction

It is extremely worthwhile to reconsider the meaning of continuity, countability, uncountability, length, and points.

Continuity is a concept that applies to the vicinity of a point. In a continuous function, the points neighbor each other with no difference no matter how much the vicinity is magnified, and the following holds:

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad (1)$$

Although the left-hand side of Equation (1) is clearly a dynamic number, the right-hand side is clearly a static number. Therefore, it is difficult to say that Equation (1) holds in the strict sense. In this paper, we adhere to the previous concept represented in Equation (1).
Countability is a concept that states that it is possible to create a 1-to-1 correspondence between the elements of the given countable set and the elements in the set of natural numbers. In contrast, uncountability is a concept that states that it is not possible to create a 1-to-1 correspondence between the elements of the given uncountable set and the elements in the set of natural numbers. It implies that it is not possible to count the elements in the set.

In addition, length refers to the distance from a starting point to an ending point. However, the question of where the starting point and the ending point are located still remains. Let us consider a given length on the number line. Let $a < b$, and let the starting point on the number line be a, and the ending point be b. Then, the length L is equal to the distance d between the starting point a and the ending point b. This can be expressed as $L = b - a$. Is this length strict? No. The reason for this answer is that the details regarding point a and point b have not been given. In general, length must refer to a half-open interval. In other words, it must refer to the half-open interval $[a,b)$ or the half-open interval $(a,b]$. The latter should be used as the appropriate half-open interval for representing the length. If a half-open interval is not chosen to represent length, then for example, the individual lengths of the closed interval [a,b] divided into 10 equal pieces would not be equal to 1/10 of the original length. Therefore, we choose a half-open interval $(a,b]$ to represent length.

In this situation, you may notice that the borderline at the immediate vicinity of point a in the half-open interval $(a,b]$ is blurry. In other words, although the interval length L of a half-open interval $(a,b]$ can be represented as $L = b - a$ using the previous formula, it can be seen that the interval length is not actually a single length. This is because no matter how closely the left end of the half-open interval $(a,b]$ approaches point a, it never reaches point a. Of course, it is acceptable to say that the limit is a. However, in the original situation, the end never reaches a.

Before we conclude this introduction, we will consider the concept of points. Euclidean points are almost identical to points in modern mathematics and refer to almost the same concept. In other words, a point is a concept that is used to accurately designate a location in space. A point is a concept that does not possess any characteristics other than location, such as length, area, volume, direction, or slope.

Based on the previous discussion, below, we consider a line segment with mysterious characteristics referred to as a discrete continuous point cloud.

1. Method for creating parallel discrete continuous lines

Let parallel lines be created by repeating the following operations an infinite number of times.

① Draw a line segment for the half-open interval (0,1] from 0 to 1 on the x-axis in x-y coordinates.
② Cut out the half-open interval (1/3,2/3] of length 1/3 from the center of the line segment in ①. Move the segment parallel upwards in the direction of the y-axis for the specified distance d.
③ Cut out similar half-open intervals of length 1/3 from the centers of each of the divided line segments that were formed. Move the divided segments that were created on the x-axis upwards in the direction of the y-axis for a distance of d, and move the line segment at $y = d$ downwards in the direction of the y-axis for a distance of d.
④Repeat operation ③an infinite number of times.
By performing operations ① through ④ above, it is possible to obtain parallel dotted lines.

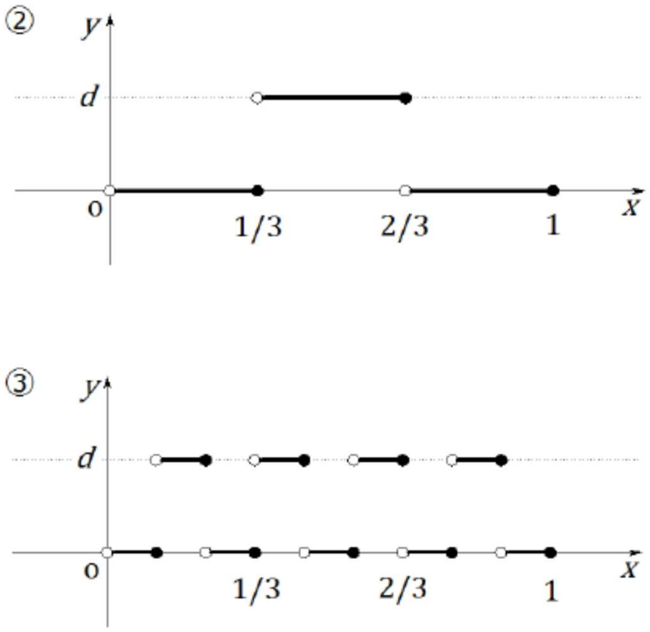

Fig. 1: Procedure for creating parallel discrete continuous dotted lines by dividing line segments

2. Let us explore the nature of these parallel dotted lines while analyzing them.

① The right ends of the divided line segments are never connected once they have been cut off.

② The right ends of the divided line segments are never cut again once they have been cut off.

③ The divided line segments, including the right ends, are never moved again once they have been moved in the direction of *y*-axis once.

④ Each divided line segment becomes shorter towards the right end (the fixed point).

⑤ The length of each divided line segment when divided for the nth time is $1/3^n$ (n = 0,1,2,3...).

⑥ The total number of divided line segments when the segment is divided for the nth time is 3^n (n = 0,1,2,3...).

⑦ The number of divided line segments at $y = 0$ is one more than the number of divided line segments at $y = d$.

⑧ The point at the right end of each line segment is a rational number point (related to 1/3), according to the Dedekind cut theorem.

⑨ The point at the left end of each line segment is not a rational number point, according to the Dedekind cut theorem.

⑩ The total length L_0 of the limit divided line segment at $y = 0$ can be represented as shown in the following equation.

$$L_0 = 1 - \frac{1}{3^1} + \frac{1}{3^2} - \frac{2}{3^2} + \frac{4}{3^3} - \frac{5}{3^3} + \cdots + \frac{\sum_{k=0}^{n-2} 3^k}{3^n} - \frac{1 + \sum_{k=0}^{n-2} 3^k}{3^n} + \cdots = \frac{1}{2}$$

⑪ The total length L_d of the limit divided line segment at $y = d$ can be represented as shown in the following equation.

$$L_d = \frac{1}{3^1} - \frac{1}{3^2} + \frac{2}{3^2} - \frac{4}{3^3} + \frac{5}{3^3} - \cdots - \frac{\sum_{k=0}^{n-2} 3^k}{3^n} + \frac{1 + \sum_{k=0}^{n-2} 3^k}{3^n} + \cdots = \frac{1}{2}$$

⑫ The sum of the total length L_0 of the limit divided line segment at $y = 0$ and the total length L_d of the limit divided line segment at $y = d$ can be represented as shown in the following equation.

$$L = L_0 + L_d = \frac{1}{2} + \frac{1}{2} = 1$$

⑬ Even though both of these parallel dotted lines exist over the interval from 0 to 1 of length 1 (especially $y = 0$), the total length of each line is only 1/2. Rather, we should say that the length is already 1/2.

⑭ Even though both of these parallel lines are dotted lines, the total length of each line is 1/2. However, Euclidean points do not have size. Therefore, if it is assumed that each element in these dotted lines is a point, then each point must be the rational point at the right end. However, there is no 1-to-1 correspondence between rational points and a continuous body, and the total sum of the sizes of the rational points does not reach 1/2. Therefore, the points in the dotted line are not Euclidean points.

⑮ The total sum length of the dotted line is 1/2 for an interval length of 1 and is an uncountable infinite length. However, it is filled with holes in every location, and the part for each hole exists on the opposite parallel dotted line. The number of points in each of the dotted lines is countable and infinite, and does not exceed the number of rational number points that are included in the half-open interval (0,1].

⑯ Within these dotted lines, there exists at most only a countable infinite number of limit divided line segments. Therefore, according to Cantor's theorem, the dotted line does not have 1-to-1 correspondence with a continuous body.

⑰ For example, the interval for the divided line segment that contains the rational point 1/3 when it is divided for the n^{th} time is

$$\left(\frac{1}{3}-\frac{1}{3^n},\frac{1}{3}\right]$$

The interval for the divided line segment when it is divided for the infinitieth time is

$$\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{3^n},\frac{1}{3}\right]=\left(\frac{1}{3}-\hat{0},\frac{1}{3}\right]\equiv\left\{\frac{1}{3}\right]$$

Therefore, it can be seen that the half-open interval becomes a half-open point. This is because one side of the interval is open, which is fundamentally the same as saying that the open side has a width of $\hat{0}$ with respect to the closed side in the direction of the x-axis. Note that $\hat{0}$ is an infinitesimally small dynamic number that represents a number that is smaller than any positive constant and is larger than 0.

Note that if we let

$$\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{3^n},\frac{1}{3}\right]=\left(\frac{1}{3}-0,\frac{1}{3}\right]=\left(\frac{1}{3},\frac{1}{3}\right]$$

then the left side of the interval in the right-hand side of the equation, which is open, does not include 1/3. On the other hand, the right side of the interval, which is closed, includes 1/3, which is a contradiction.

⑱ The interval length |{1/3] of the half-open point {1/3] is given by the following equation.

$$\lim_{n\to\infty}\left|\left(\frac{1}{3}-\frac{1}{3^n},\frac{1}{3}\right]\right|=\left|\left(\frac{1}{3}-\hat{0},\frac{1}{3}\right]\right|=\frac{1}{3}-\left(\frac{1}{3}-\hat{0}\right)=\hat{0}>0$$

$$\therefore\quad\left|\left\{\frac{1}{3}\right]\right|=\hat{0}>0$$

⑲ The individual half-open points in the limit divided line segment satisfy Equation (1), and are therefore continuous and have a 1-to-1 correspondence with a continuous body.

⑳ The parallel lines that were formed within the interval of length 1 by a dense collection of alternating locations where the line exists and does not exist (half-open points) do not have a 1-to-1 correspondence with a continuous body as line segments. However, the individual half-open points in the divided line segment do have a 1-to-1 correspondence with a continuous body.

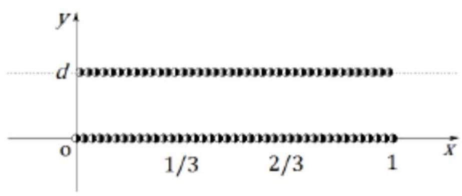

Fig. 2: An image of parallel discrete continuous dotted lines when divided to the limit.

Based on the analysis shown above, it can be said that four types of points exist: closed points [a], left half-open points (a], right half-open points [a), and open points (a). Furthermore, the sizes of these points are given by:

$$|[a]| = 0,\ \ 0 < |\{a]| < \varepsilon,\ \ 0 < |[a\}| < \varepsilon,\ \ 0 < |(a)| < \varepsilon$$

Therefore, this implies that Euclidean points are closed points. Based on the above result, the following theorem immediately holds.

Theorem: Dividing a continuous body to the limit does not result in each divided body consisting of a closed point.

Furthermore, this implies that the reverse proposition, which states that “it is not possible to form a continuous body with closed points,” is true. Simultaneously, this also strongly implies that the greatest proposition within the history of the field of mathematics, which states that “lines are made out of points,” is false.