February 17, 2016
Hiroshi Michiwaki

The zero logarithm log0 = 0 and division by zero

Theorem: For the logarithm with a base equal to Napier's constant *e*,

$$\log 0 = 0$$

holds.

Proof: Consider the definite integral of the hyperbolic function $y = 1/x$ for x over the closed interval [0,0]. In other words, consider

$$\int_{[0}^{0]} \mathrm{d}x \frac{1}{x} \qquad (1)$$

According to the fundamental theorem of division by zero, 0/0 = 0, this integral becomes

$$\int_{[0}^{0]} \mathrm{d}x \frac{1}{x} = \left[\frac{1}{x} \cdot x\right]_{x=0} = \frac{0}{0} = 0 \qquad (2)$$

Conversely, according to the definite integral of the hyperbolic function, this becomes

$$\int_{[0}^{0]} \mathrm{d}x \frac{1}{x} = [\log_e x]_{[0}^{0]} = \log_e 0 - \log_e 0 \qquad (3)$$

Based on Equation (2) and Equation (3),

$$\log_e 0 - \log_e 0 = 0 \qquad (4)$$

holds. Here, the left-hand side of Equation (4) can be transformed into

$$\log_e 0 - \log_e 0 = \log_e 0 + \log_e 0^{-1} = \log_e 0 + \log_e 0 \qquad (5)$$

by applying division by zero $0^{-1} = 0$. Therefore, based on Equation (4) and Equation (5),

$$\log_e 0 + \log_e 0 = 0 \qquad (6)$$

holds. Based on this, we immediately obtain

$$2\log_e 0 = 0$$
$$\therefore \quad \log_e 0 = 0 \qquad (7)$$