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Infinity, the Infinity Point, and Division by Zero

Given a real number x,

$$\lim_{x\to\pm 0}\frac{1}{x}=\pm\infty \qquad (1)$$

based on the fundamental theorem of division by zero, $z/0=0$, upon reaching the absolute point where $x=$ 0, rather than getting arbitrarily close to 0 as in $x\to\pm 0$,

$$\left[\!\left[\frac{1}{x}\right]\!\right]_{x=0}=0 \qquad (2)$$

This is equivalent to the infinity point 1/0 corresponding to 0. This also shows that

$$\lim_{x\to\pm\infty} x=\pm\infty \qquad (3)$$

and upon reaching the absolute point where x = [±∞], rather than getting arbitrarily close to ±∞ as in x = ±∞,

$$[\![x]\!]_{x=[\pm\infty]}=0 \qquad (4)$$

It should be noted that here, [±∞] denotes the infinity point; that is, the absolute value increases without limit, with brackets to distinguish it from ±∞, which represents infinity.

Considering the above,

Theorem 1 For a>1, given

$$\lim_{n\to+\infty} a^n=+\infty$$

the following is true:

$$[\![a^n]\!]_{n=[+\infty]}=0$$

Theorem 2 For a>1, given

$$\lim_{n\to\pm\infty}\frac{1}{a^n}=0$$

the following is true:

$$\left[\!\left[\frac{1}{a^n}\right]\!\right]_{n=[\pm\infty]}=0$$

$$\because \quad \left[\!\left[\frac{1}{a^n}\right]\!\right]_{n=[+\infty]} = \frac{[\![1]\!]}{[\![a^n]\!]_{n=[+\infty]}} = \frac{1}{0} = 0$$

where $n = [-\infty]$ is equivalent to Theorem 1. QED

This immediately implies the following.

Corollary 1 In the limit as $n \to +\infty$, the sum contains a term without a limit but does not reach the infinity point. As such,

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{9}{10^k} = 0.\dot{9} \neq 1$$

However, if the sum is up to the infinity point $n = [+\infty]$, then by Theorem 2,

$$\left[\!\left[\sum_{k=1}^{n} \frac{9}{10^k}\right]\!\right]_{n=[+\infty]} = \left[\!\left[1 - \frac{1}{10^n}\right]\!\right]_{n=[+\infty]} = 1 - \left[\!\left[\frac{1}{10^n}\right]\!\right]_{n=[+\infty]} = 1 - 0 = 1$$

which shows that it equals exactly 1.

Therefore, the following is true:

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{9}{10^k} < \left[\!\left[\sum_{k=1}^{n} \frac{9}{10^k}\right]\!\right]_{n=[+\infty]}$$

Corollary 2 Given an infinitesimally small positive number ε greater than 0 and smaller than any positive constant δ ($0 < \varepsilon < \delta$) no matter how small,

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to+\infty} \left(1 - \frac{1}{2^n}\right) = 1 - \varepsilon < 1$$

However, if the sum is up to the infinity point $n = [+\infty]$, then the following is true:

$$\left[\!\left[\sum_{k=1}^{n} \frac{1}{2^k}\right]\!\right]_{n=[+\infty]} = \left[\!\left[1 - \frac{1}{2^n}\right]\!\right]_{n=[+\infty]} = 1 - \left[\!\left[\frac{1}{2^n}\right]\!\right]_{n=[+\infty]} = 1 - 0 = 1$$

Pot of Infinitely Many Numbers

How can one express a number s that is smaller than 1 yet greater than any constant smaller than 1? Expressing this with a single number requires an infinite decimal because the following would be true if s were a finite decimal:

$$0 < 0.9 < 0.99 < 0.99 \cdots 9 < s \leq \frac{9+s}{10} < 1 \qquad (1)$$

which would clearly imply that

$$s = 0.999 \cdots \qquad (2)$$

s is an infinite decimal. (This can also be abbreviated as $s = 0.\dot{9}$.)

However, according to modern mathematics,

$$0.999 \cdots = 1 \qquad (3)$$

Is this actually correct? The representative example of the basis for this derivation is to multiply both sides of (2) by 10 and then subtract both sides of (2) from the resulting equation, namely

$$\begin{array}{rl} 10s = 9.999 \cdots & \\ -) \quad s = 0.999 \cdots & \\ \hline 9s = 9.000 \cdots & (4) \end{array}$$

Dividing both sides by 9 would then lead to the result s=1.

However, is this operation strictly mathematically correct? s is supposed to be smaller than 1 yet greater than any constant smaller than 1, which is a contradiction. What is the cause of such a contradiction? It may lie in the fact that this operation treats the infinite series of a limit as if it were a finite series. Can it be stated that the infinitely small terms that continue indefinitely were treated in some clever manner so that the difference between them was 0 after the subtraction?

Strictly speaking, s is an infinite series, expressed thus:

$$s = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{9}{10^k} \qquad (5)$$

As this is a limit, the partial sum becomes arbitrarily close to 1 but never reaches 1. Generally, in infinite series changing the order of terms or performing operations term by term is not permitted. That is, is it true that multiplications performed term by term, as in (4), or decimal place by decimal place yield the expected result in the limit?

To answer this question, the problem should be considered from a different perspective. Here is a thought experiment involving a mysterious pot that represents a particular number. If you reach into the pot, you can pull out the digit at each decimal place of this number indefinitely. Even if you do not, the pot still contains the source of each digit. The digit at the first decimal place is 9, and so is the second, the third, and so on, so that you can keep on taking out 9s at all successive decimal places.

The following is a visualization of this pot:

$$S=0.99999\cdots 9^{9^{9_9}}\mho$$

Then, would it be possible to perform an operation such as that in (4), that is, multiplying both sides by 10? Without going through every number in the pot, this may be carried out as follows:.

$$\begin{array}{rl} & 10S=9.9999+10\mho_5 \\ -) & S=0.99999+\ \ \mho_5 \\ \hline & 9S=8.99991+9\mho_5 \end{array}$$

Dividing both sides by 9 naturally yields

$$S=0.99999+\mho_5$$

Here, the subscript number 5 on the pot represents the fact that the digits corresponding to five decimal places were taken out of the pot. Below, a subscript n on a pot will denote that there are digits corresponding to n places in the pot.

The result of this calculation suggests that extending the multiplication of each decimal place to infinity is probably not valid.

This implies that based on this thought experiment with the mysterious pot, multiplying both sides by 10 simply yields 10S on the left-hand side, and a 5-decimal-place number and ten mysterious pots on the right-hand side. Furthermore, the difference between a pot with n+1 digits and n digits is expressed as

$$\mho_{n+1}-\mho_n=9/10^{n+1}$$

That is, although letting n tend to the limit will cause the difference to get arbitrarily close to 0, this does not imply that the pot will cease to exist at that point; rather, for all intents and purposes, this difference will never be 0.