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Hiroshi Michiwaki

## Expanding Compound Fractions Using Division by Zero

**Theorem** Given two nonnegative integers $n$, $m$, when

$$n-0\times m>n-1\times m>n-2\times m>\cdots>n-k\times m=r \quad (k\in \mathrm{N}_0 : \text{Nonnegative integers}) \qquad (1)$$

and $r$ is the smallest nonnegative integer that satisfies (1),

$$\frac{n}{m}=0\frac{n-0\times m}{m}=1\frac{n-1\times m}{m}=2\frac{n-2\times m}{m}=\cdots=k\frac{n-k\times m}{m}=k\frac{r}{m}=k+\frac{r}{m}$$

$$\Rightarrow \frac{n}{m}=k\cdots r \quad \wedge \quad n=km+r \qquad (2)$$

where $\cdots r$ denotes the remainder obtained after dividing $n$ by $m$.

**Proof** When $m\neq 0$,

**i.** $n>m$
If $n=km+r$ ($k\in\aleph_0 \wedge 0\leqq r<m$), it is clear.

**ii.** $n=m$
In (1),
$n-0\times m>n-1\times m=0 \quad (k\in \mathrm{N}_0 : \text{Nonnegative integers})$
implies that $k=1\wedge r=0$. This yields
$\frac{n}{m}=1\cdots 0 \quad \wedge \quad n=1\times m+0=m$

**iii.** $n<m$
In (1),
$n-0\times m=n \quad (k\in \mathrm{N}_0 : \text{Nonnegative integers})$
implies that $=0\wedge r=n$. This yields
$\frac{n}{m}=0\cdots n \quad \wedge \quad n=0\times m+n=n$

**iv.** $n=0$
In (1),
$n-0\times m=n \quad (k\in \mathrm{N}_0 : \text{Nonnegative integers})$
implies that $k=0\wedge r=n=0$. That yields
$\frac{n}{m}=0\cdots 0 \quad \wedge \quad n=0\times m+0=0$

When $m=0$,

In (1),

$$n-k\times 0=n \quad (k\in \mathrm{N}_0 : \text{Nonnegative integers})$$

implies that $k=0\wedge r=n$. Therefore,

$$\frac{n}{m}=\frac{n}{0}=0\frac{n-0\times 0}{0}=0\frac{r}{0}=0+\frac{r}{0}=\frac{n}{0}$$

Consequently,

$$\Rightarrow \frac{n}{0} = 0 \cdots n \quad \wedge \quad n = 0 \times 0 + r = 0 \times 0 + n = n$$

QED.

It can be seen from the above that it is possible to expand a compound fraction in a natural form by letting the denominator (divisor) equal 0.

Moreover, in the relation

$$k\frac{r}{m} = k + \frac{r}{m} \qquad (3)$$

if $m = 0$ and $k \neq 0$, it should be noted that

$$k + \frac{r}{m} = k\frac{r}{m} \qquad (4)$$

cannot lead to

$$\frac{n}{m} = k \cdots r \quad \wedge \quad n = km + r \qquad (5)$$

The reason for this is that (4)

$$k + \frac{r}{m} = k\frac{r}{m} = \frac{r + km}{m} = \frac{r}{m} + \frac{km}{m} = \frac{r}{m} + \frac{m}{m}k$$

$$\therefore \quad k = \frac{m}{m}k$$

implies that

$$\frac{m}{m} = 1 \qquad (6)$$

However, by Theorem,

$$\frac{0}{0} = 0 \qquad (7)$$

which is inconsistent. That is, it should be noted that the zero-denominator fraction that is obtained when $m = 0$ cannot be reduced to a nonzero denominator.