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Density Principle and Division by Zero with the Resulting Remainder

Let us assume in a space with a uniform gravitational acceleration g, there is a cylinder with radius R, height h, uniform density ρ, and total mass M, positioned on a level, circular pressure receptor A_0 with total surface radius $r_0 \geqq R$.

The total force of gravity F acting on the total mass M in this spatial system is then expressed as

$$F = Mg \qquad (1)$$

Inside the receptor A_0, a pressure receptor $A \leqq A_0$ is considered, which will be the target of the discussion. This surface A, which has radius r, is flat and concentric to A_0. If now M_{out} is used to represent the mass of the cylinder that is not on the target surface A, the total force of gravity F, based on Equation (1), is

$$Mg = ps + M_{out}g \qquad (2)$$

where p is the pressure acting on the target surface A and s is its area, expressed as $s(r)=\pi r^2$. Consequently, if M_{in} is used to represent the mass of the cylinder that *is* on the target surface A, the pressure p can be obtained by

$$p = \frac{M_{in}g}{\pi r^2} \qquad (3)$$

Based on this, eliminating g by substituting (3) into (2) yields

$$M = \frac{M_{in}}{\pi r^2}s + M_{out} \qquad (4)$$

which can also be expressed as

$$\frac{M}{s} = \frac{M_{in}}{\pi r^2} + \frac{M_{out}}{s} \qquad (5)$$

Alternatively, if $\sigma(r)$ is used to represent the mass surface density on the target surface A, (4) can be expressed as

$$M = \sigma(r)s + M_{out} \qquad (6)$$

If (5) and (6) are considered from the perspective of division, the total mass M can be regarded as the dividend, the mass surface density σ as the quotient, the area s as the divisor, and the mass outside the target surface, M_{out}, as the remainder. That is, (6) can be expressed as

$$\frac{M}{s} = \sigma(r) \cdots M_{out} \qquad (7)$$

Here, $\cdots$ indicates that the M_{out} written after the $\cdots$ is the remainder of the division operation.

The effect of a reduction in radius r to the target surface A if r_0 is fixed will now be examined. That is, the conditions of the system remain unchanged except for the size of surface A, and then Equations (4)–(7) are considered.

The mass surface density $\sigma(r)$ then is

$$\sigma(r) = \frac{M_{in}}{\pi r^2} = \frac{\rho\pi r^2 h}{\pi r^2} = \rho h \quad (r > 0) \qquad (8)$$

and as r gradually decreases, this causes the part of the cylinder that is not on the target surface A to expand. Consequently, the mass outside the target surface, M_{out}, is expressed as

$$M_{out} = \rho\pi(R^2 - r^2)h \qquad (9)$$

Combining (6), (8), and (9) yields

$$\begin{aligned} M &= M_{in}(r) + M_{out}(r) \\ &= \rho h s + \rho\pi(R^2 - r^2)h \\ &= \rho\pi r^2 h + \rho\pi(R^2 - r^2)h \\ &= \rho\pi R^2 h \qquad (r > 0) \quad (10) \end{aligned}$$

Equation (10) represents the transitional mechanism of the mass corresponding to the radius r of the area s of the surface A. If now $r = 0$, the surface area s of A will also be 0, and thus the surface becomes a dimensionless point. Hence, the volume of the cylinder on the target surface A, $v = \pi r^2 h$, is also zero. Consequently, the mass atop a point with an area of 0 is necessarily $M_{in}(r) = \rho v(r) = \rho v(0) = 0$. It can also be claimed that a surface with area a=0 is a point and thus a nonexistent area (null φ = 0). The mass m atop the point—that is, the nonexistent area (null φ)—is also m=0, which makes area a and mass m both 0 (null φ) and nonexistent. This implies that the mass surface density σ is also nonexistent (null φ) and thus $\sigma = 0$. Therefore, it is clear that

$$\sigma(0) = 0 \qquad (11)$$

The following will be referred to as the mass surface density principle (or simply the "density principle," as it can also be applied to other densities as well):

$$\sigma(r)s = \sigma(r)s(r) = \sigma(0)s(0) = 0 \times 0 = 0 \qquad (12)$$

This shows that the first term in (6) would then be 0. Moreover, as clearly seen from (9), the second term in (6) would become

$$M_{out} = M_{out}(r) = M_{out}(0) = \rho\pi(R^2 - 0^2)h = \rho\pi R^2 h = M \qquad (13)$$

That is, (6) implies

$$M = 0 \times 0 + M_{out} \qquad (14)$$

If, as in (7) above, the total mass M is regarded as the dividend, the mass surface density σ as the quotient, the area s as the divisor, and the mass outside the target surface, M_{out}, as the remainder, then (14) yields

$$\frac{M}{0} = 0 \cdots M_{out} \quad (M_{out} = M) \qquad (15)$$

where $\cdots$ denotes that M_{out} is the remainder of the division operation. Consequently, this arguably demonstrates that this physical system can be expressed mathematically by using division by zero with a remainder. Conversely, it arguably shows that such a method is mathematically correct for this physical system as well. Below, Equation (15) will be referred to as the First Law of Mass Surface Density.

The case where r_0 (the radius of the entire level pressure receptor A_0 in this system) atop which the cylinder is placed becomes $r_0 < R$ with respect to R (the radius of the cylinder) is now considered. That is, when $A = A_0$ $(r = r_0)$. Naturally, the total force of gravity F acting on the total mass M in this spatial system is expressed as

$$F = Mg \qquad (16)$$

Thus, if $A=A_0$, then the level circular pressure receptor A_0 with radius r_0 is the now target surface A and its radius is r. As the mass in this system is concentrated on target surface A, if M_{off} denotes the mass not sitting on target surface A, the total force of gravity F is expressed as

$$Mg = ps + M_{off}g \qquad (17)$$

by (16), where p is the pressure acting on target surface A and s is its area, expressed as $s(r) = \pi r^2$. Consequently, if M_{on} denotes the mass on target surface A, p can be determined by

$$p = \frac{M_{on}g}{\pi r^2} \qquad (18)$$

Based on this, eliminating g by substituting (18) into (17) yields

$$M = \frac{M_{on}}{\pi r^2}s + M_{off} \qquad (19)$$

which can also be expressed as

$$\frac{M}{s} = \frac{M_{on}}{\pi r^2} + \frac{M_{off}}{s} \qquad (20)$$

Alternatively, if $\sigma(r)$ represents the mass surface density on the target surface A, (19) can be expressed as

$$M = \sigma(r)s + M_{off} \qquad (21)$$

Considering (20) and (21) from the perspective of division, the total mass M can be regarded as the dividend, the mass surface density σ as the quotient, the area s as the divisor, and the mass outside the target surface, M_{out}, as the remainder. That is, (21) can be expressed as

$$\frac{M}{s} = \sigma(r) \cdots M_{off} \qquad (22)$$

Here, $\cdots$ indicates that M_{out} written after the $\cdots$ is the remainder of the division operation.

The effect of a reduction in the area of both the target surface A and the surface A_0 will now be examined so that that A_0 becomes smaller than the bottom surface of the cylinder with radius R. That is, the size and shape of the cylinder remain unchanged, and only the size of surface A (= A_0) is varied, and then (19)–(22) are considered. Of course, there will be some changes in the system, as the target surface A, which is the entire pressure receptor, is changed, but the total mass M will evidently remain unchanged.

The mass surface density $\sigma(r)$ then is

$$\sigma(r) = \frac{M_{on}}{\pi r^2} = \frac{\rho\pi R^2 h}{\pi r^2} = \rho R^2 h \frac{1}{r^2} \quad (r > 0) \qquad (23)$$

and as r gradually decreases, the mass surface density $\sigma(r)$ on the target surface A increases rapidly, in inverse proportion to r^2. Furthermore, based on (21) and (23), the total mass M can be expressed as

$$\begin{aligned} M &= \rho R^2 h \frac{1}{r^2} s + M_{off}(r) \\ &= \rho\pi R^2 h + M_{off}(r) \\ &= M_{on}(r) + M_{off}(r) \qquad (r > 0) \quad (24) \end{aligned}$$

Equation (24) represents the transitional mechanism of the mass corresponding to the area (r) of the surface A of this system. Moreover, as A occupies a finite area, the entire mass M is sitting on A, which implies that M_{off}, the mass not sitting on A, is

$$M_{off} = 0 \qquad (r > 0) \qquad (25)$$

Here, if $r = 0$ for the target surface A, its area will then be (0) = 0, and the surface will become a dimensionless point. Then, according to the mass surface density principle, $M_{on}(r)$, the mass sitting on the area $s(0)$ of this dimensionless point (target surface A), is nonexistent. (This fact is self-evident, as it is not possible to place a mass on a nonexistent area.) Accordingly,

$$M_{on}(r) = M_{on}(0) = 0 \qquad (26)$$

and

$$\sigma(0) = 0 \qquad (27)$$

(It can be stated that the pressure acting on a surface with no area is $p = \sigma g = \sigma(0)g = 0 \mathrm{x} g = 0$.) Consequently, the first term in (21) then becomes

$$\sigma(r)s = \sigma(r)s(r) = \sigma(0)s(0) = 0 \times 0 = 0 \qquad (28)$$

Moreover, based on the definition of M_{off} as the mass not sitting on the target surface A, the second term in (21) is

$$M_{off} = M_{off}(r) = M_{off}(0) = \rho \pi R^2 h = M \qquad (29)$$

Thus, (21) is

$$M = 0 \times 0 + M_{off} \qquad (30)$$

where, as above, the total mass M can be regarded as the dividend, the mass surface density σ as the quotient, the area s as the divisor, and the mass outside of the target surface, M_{off}, as the remainder. Therefore,

$$\frac{M}{0} = 0 \cdots M_{off} \quad (M_{off} = M) \qquad (31)$$

(where $\cdots$ denotes that M_{off} is the remainder of the division operation), which implies that this system can be described mathematically by using division by zero with a remainder.

Of course, multiplying both sides of (30) by gravitational acceleration g yields

$$F = Mg = (0 \times 0 + M_{off})g = 0 \times g + M_{off}g = M_{off}g \qquad (32)$$

which implies that the force of gravity acting on the zero-area (point) of the target surface is zero and that the force exerted by the mass outside the target surface area is equal to the total force F; furthermore, without any part of the mass supported by the surface, the force of gravity will cause the total mass to drop.

Thus, it can be claimed that

1. Phenomena such as mass concentrating on a point whose size is zero will not occur.
2. The surface of a table upon which a cup is placed has an uncountably infinite number of points, and the surface pressure at each of these points is zero.

That is, it is clear that the values, for instance, for mass surface density, surface pressure, and force of gravity on a zero-area surface (a point) are constantly zero, without becoming infinitely large, diverging, becoming undefined, and taking any value. Moreover, the mass atop the point never becomes infinitely large, the total mass never becomes zero, and the force of gravity acting on the point never becomes infinitely large. Furthermore, the mass surface densities $\sigma(r)$ shown in (8) and (23) are

$$\sigma(r) = \frac{M_{in}}{\pi r^2} = \frac{\rho \pi r^2 h}{\pi r^2} = \rho h \left(\frac{r}{r}\right)^2 = \begin{cases} 0 & (r = 0) \\ \rho\mathrm{h} & (r > 0) \end{cases} \qquad (8')$$

$$\sigma(r) = \frac{M_{on}}{\pi r^2} = \frac{\rho \pi R^2 h}{\pi r^2} = \rho h \left(\frac{R}{r}\right)^2 = \begin{cases} 0 & (r = 0) \\ \rho h \left(\frac{R}{r}\right)^2 & (r > 0) \end{cases} \qquad (23')$$

respectively, which in turn yield

$$\sigma(r) \begin{cases} = 0 & (r = 0) \\ > 0 & (r > 0) \end{cases} \qquad (33)$$