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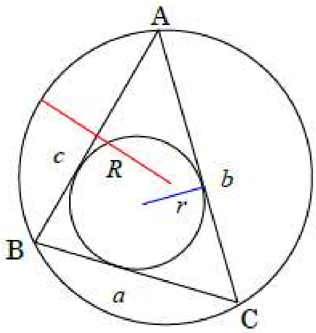
Division by Zero, as Illustrated by the Inscribed and Circumscribed Circles of a Triangle

Let r and R be the radii, respectively, of the inscribed and circumscribed circles of a triangle ΔABC with sides a, b, c opposite to vertexes A, B, C, as in the figure below.



Then, the area S of △ABC is

$$S = \frac{r}{2}(a+b+c) = \frac{abc}{4R} \qquad (1)$$

It is now assumed that ΔABC is shrunk beyond its limit, that is, the radius R of the circumscribed circle is $R = 0$. Of course, this implies that the radius r of the inscribed circle would also be zero. Consequently, the middle part of (1) would be

$$\frac{r}{2}(a+b+c) = \frac{0}{2}(0+0+0) = 0 \qquad (2)$$

and the right side would be

$$\frac{abc}{4R} = \frac{0\times 0\times 0}{4\times 0} = \frac{0}{0} \qquad (3)$$

Consequently, the left, middle, and right sides of (1) all represent the area of ΔABC, and it is self-evident that the area S of ΔABC is 0 when the radius R of the circumscribed circle is 0. Thus, (2) and (3) yield

$$\frac{0}{0} = 0 \qquad (4)$$

Note Let the inverse relational expression of (1) be considered, namely

$$\frac{1}{S} = \frac{2}{r(a+b+c)} = \frac{4R}{abc} \qquad (5)$$

When $R = 0$, (5) is

$$\frac{1}{0} = \frac{2}{0 \times (0 + 0 + 0)} = \frac{0}{0} = 0 \qquad (6)$$

by (4), yielding

$$\frac{1}{0} = 0 \qquad (7)$$

It is now assumed that ΔABC is enlarged beyond its limit, that is, the radius R of the circumscribed circle is $R = \dot{\infty}$. Then, the middle part of (5) becomes

$$\frac{2}{\dot{\infty}(\dot{\infty} + \dot{\infty} + \dot{\infty})} = \frac{2}{\dot{\infty}} = 0 \qquad (8)$$

and the right side becomes

$$\frac{4 \times \dot{\infty}}{\dot{\infty} \times \dot{\infty} \times \dot{\infty}} = \frac{\dot{\infty}}{\dot{\infty}} \qquad (9)$$

Thus, (8) and (9) yield

$$\frac{\dot{\infty}}{\dot{\infty}} = 0 \qquad (10)$$

Furthermore, by (5) and (10),

$$\frac{1}{\dot{\infty}} = \frac{\dot{\infty}}{\dot{\infty}} = 0 \qquad (11)$$

and considering (7), we obtain

$$\dot{\infty} = 0 \qquad (12)$$

The symbol $\dot{\infty}$ here denotes true infinity, which is a point beyond infinity or the size between the origin and the point of true infinity. That is, a triangle with a circumscribed circle (inscribed circle) whose radius is true infinity can be considered to have an area equal to zero.