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Reducible set theoretical division of complex numbers

Theorem 1: In the division of two complex numbers, z_1 and z_2, z_1/z_2, there is a complex quotient z_3 and a complex remainder z_4 that satisfy

$$\frac{z_1}{z_2} = z_3 \ldots z_4$$

where "…" means that the value immediately after "…" is the remainder term.

Theorem 2: In the division of two complex numbers, z_1 and z_2, z_1/z_2, if we define the quotient of reducible set theoretical division of real numbers $|\{r2\}|$ using the complex radius r_2 of z_2 as the smallest non-negative real number when $|\{r2\}|$ is maximized,

$$\frac{z_1}{z_2} = |\{r_2\}| \cdot \{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \ldots s\{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\}$$

is established. However, θ_1 and θ_2 are real numbers, and are the complex azimuth of z_1 and z_2, respectively. "…" means that the value immediately after "…" is the remainder term.

Theorem 3: When $z \in$ C,

$$\frac{z}{0} = 0 \ldots z$$

holds.

Proof: Proofs for Theorems 1, 2, and 3 are shown below without separation.

If we express the complex number $z = a + bi$ (a, b $\in$ R, i: imaginary unit) using Euler's formula as a polar form,

$$z = re^{i\theta} = (\cos\theta - i\sin\theta) \quad (1)$$

where r is the distance from the origin on the complex plane (complex radius), and θ is the slope angle from the positive side of the real axis. Both r and θ are real numbers. Using these values, two complex numbers z_1 and z_2 are expressed as

$$z_1 = r_1 e^{i\theta 1} \quad (2)$$
$$z_2 = r_2 e^{i\theta 2} \quad (3)$$

If z_1 is divided by z_2, it is expressed as

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \qquad (4)$$

The result of Equation (4) is a complex number with a complex radius of r_1/r_2 that occurs as a result of division z_1/z_2, and the term inside of { } is the unit radial azimuth component.

Definition of reducible set theoretical division of real numbers

When *A* and *B* are real numbers, the calculation of division *B*/*A* is as follows. In

$$|B| - (|\{A\}| \cdot |A| + a) = 0 \ (0 \leq a) \ (A, B, a \in \mathrm{R}) \ (5)$$

we assume $B_0 = B$ and $A_j = A$ (j = 0, 1, 2, …), and define a set based on A_{j+1} (this A_{j+1} is called the j+1th reducible number) in the reducible recurrence relation $B_j - A_{j+1} = B_{j+1}$ as the j+1th reducible number set $\{A_{j+1}\}$, and when B_j (this B_j is called the j^{th} reduced number) satisfies $B_j > B_{j+1} \geq 0$, $\{A_{j+1}\} \neq \emptyset$, and when this condition it not satisfied, $\{A_{j+1}\} = \emptyset$. The set whose elements are all within the reducible number set $\{A_{j+1}\}$ is defined as the reducible set $\{A\}$. Here, *B* is the dividend, *A* is the divisor, and $|\{A\}|$ is the number of elements of the reducible set $\{A\}$, which is the quotient of *B*/*A*, where *a* is the remainder, which is the smallest non-negative real number when $|\{A\}|$ is maximized.

Therefore, if r_1/r_2 is treated based on reducible set theory, when the remainder is ***s***, it is expressed as

$$|r1| - (|\{r2\}| \cdot | r2| + s) = 0 \ (0 \leq s) \ (r1, r2, s \in \mathrm{R}) \ (6)$$

Therefore Equation (4) becomes

$$\frac{z_1}{z_2} = |\{r_2\}| \cdot \{\cos(\theta_1 - \theta_2) - i \sin(\theta_1 - \theta_2)\} \cdots s\{\cos(\theta_1 - \theta_2) - i \sin(\theta_1 - \theta_2)\} \qquad (7)$$

Here, the quotient and remainder terms of Equation (7) are

$$z_3 = |\{r_2\}| \cdot \{\cos(\theta_1 - \theta_2) - i \sin(\theta_1 - \theta_2)\} \qquad (8)$$
$$z_4 = s\{\cos(\theta_1 - \theta_2) - i \sin(\theta_1 - \theta_2)\} \qquad (9)$$

respectively, and can be expressed as

$$\frac{z_1}{z_2} = z_3 \ldots z_4 \qquad (10)$$

Here, we can see that the quotient of Equation (10) is the complex quotient z_3, which is a complex number, and the remainder is the complex remainder z_4, which is a complex number. In other words, division of a complex number by a complex number can be defined by the reducible set theory as shown above. It questions the ratio of the complex radius of z_1 (r_1) to the complex radius of z_2 (r_2); i.e., the complex radius ratio. At the same time, it refers to the number of times the complex radius r_2 can be reduced from the complex radius r_1 of z_1, and it asks the size of the difference in the complex azimuth. The true nature of asking these two questions is that the amount of information in the original complex number z consists of two points; in other words, two sets of information on the real axis and the imaginary axis.

In Equation (7), let us suppose a case where $\theta 1 = \theta 2 = 0$. In this case, Equation (7) can be reduced as follows:

$$
\begin{aligned}
\frac{z_1}{z_2} &= |\{r_2\}| \cdot \{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \dots s\{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \\
&= |\{r_2\}| \cdot \{\cos 0 - i\sin 0\} \dots s\{\cos 0 - i\sin 0\} \\
&= |\{r_2\}| \cdot \{1 - i \cdot 0\} \dots s\{1 - i \cdot 0\} \\
&= |\{r_2\}| \cdot 1 \dots s \cdot 1 \\
&= |\{r_2\}| \dots s \\
&= \frac{r_1}{r_2} \qquad (11)
\end{aligned}
$$

Therefore, we can see that the reducible set theoretical complex division expressed in Equation (7) is a natural expansion of the reducible set theoretical division of real numbers.

Also, in Equation (7), if $r_2 = 0$, it is clear that $z_2 = 0$ based on Equation (3), and can be expressed as $\theta_2 = |\emptyset|$; therefore,

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1}{0} \\
&= |\{r_2\}| \cdot \{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \dots s\{\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)\} \\
&= 0 \cdot \{\cos(\theta_1 - |\emptyset|) - i\sin(\theta_1 - |\emptyset|)\} \dots s\{\cos(\theta_1 - |\emptyset|) - i\sin(\theta_1 - |\emptyset|)\} \\
&= 0 \dots s\{\cos(\theta_1 - 0) - i\sin(\theta_1 - 0)\} \\
&= 0 \dots s\,(\cos\theta_1 - i\sin\theta_1) \\
&= 0 \dots r_1\,(\cos\theta_1 - i\sin\theta_1) \\
&= 0 \dots z_1 \\
\therefore \quad \frac{z_1}{0} &= 0 \dots z_1 \qquad (12)
\end{aligned}
$$

is obtained.